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Finite-time stability and stabilization of semi-Markovian jump systems with time delay

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Summary

Semi-Markovian jump systems are more general than Markovian jump systems in modeling practical systems. On the other hand, the finite-time stochastic stability is also more effective than stochastic stability in practical systems. This paper focuses on the finite-time stochastic stability, exponential stochastic stability, and stabilization of semi-Markovian jump systems with time-varying delay. First, a new stability condition is presented to guarantee the finite-time stochastic stability of the system by using a new Lyapunov-Krasovskii functional combined with Wirtinger-based integral inequality. Second, the stability criterion is further proved to guarantee the exponential stochastic stability of the system. Moreover, a controller design method is also presented according to the stability criterion. Finally, an example is provided to illustrate that the proposed stability condition is less conservative than other existing results. Additionally, we use the proposed method to design a controller for a load frequency control system to illustrate the effectiveness of the method in a practical system of the proposed method.

KEYWORDS

exponential stochastic stability analysis, finite-time stability analysis, semi-Markovian jump systems, stabilization, time-delay systems

1 | INTRODUCTION

Many practical systems have structures subject to random abrupt changes in inputs, internal variables, and other system parameters, which could be caused by component failures and sudden environmental disturbances. In order to describe such kind of systems, Markovian jump linear systems (MJLSs) are firstly introduced by Krasovskii and Lidskii.[1] The

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control of MJLSs has been a hot research subject and received great attention in the past decades. Lots of results related to such systems have been reported in the literature.[2-10] However, the jump time of a Markov chain is supposed to be exponentially distributed in general, which limits the applications of MJLSs. Meanwhile, since in MJLSs, the transition rates are considered to be constants, which would definitely be more conservative than the results of semi-MJLSs (S-MJLSs) because the transition rates of S-MJLSs are supposed to be time varying. Thus, MJLSs have many limitations in applications, and the results obtained on it are conservative in some sense. Since S-MJLSs are more common than MJLSs, much attention has been paid to S-MJLSs in the literature, which has wide applications such as power systems, vehicles, and aircrafts.[11,12] The probability of the distribution of sojourn time is replaced from exponential distribution to Weibull distribution, and some significant results were presented in the work of Huang and Shi.[13] Further results on the controller design of S-MJLSs were also proposed in the work of Huang and Shi.[14] In the work of Hou et al,[15] the stability of Ito differential equations with semi-Markovian jump parameters was investigated. In the work of Zhang et al,[16] a semi-Markov kernel approach was presented to investigate the stability of discrete-time S-MJLSs.

In practical industry systems, there always exists time delay, which is an important source of instability and poor performance. In most of circumstances, the exact value of delay is not possible to be known in advance, which can only be estimated in a controller design process. Thus, the research of time-delay systems has attracted many researchers from the control community. Naturally, since the investigation of MJLSs with time delay is a very important branch, the research for S-MJLSs with time delay is a hot topic nowadays. To mention a few, in the work of Li et al,[17] the time varying transition rates were expressed as an average value and a disturbance, which make the result easy to be implemented in MATLAB. In the work of Huang and Shi,[18] the sojourn time partition technique was firstly proposed.

On the other hand, finite-time stability is a more practical concept, and it is very helpful to study the behavior of the system within a finite possible short interval.[19-22] Thus, it finds application whenever it is desired that the state variables do not exceed a given threshold during the transients. In a practical process, we usually pay a great attention to the behavior of systems in a fixed time interval. For example, in some special applications, we need the system stable error that is small enough in a fixed time interval, but some controller design methods only consider the stability of the system theoretically, which cannot satisfy the fast stability requirement. Thus, the finite-time stability is investigated to address these transient performances of control systems. For a practical network control system, it is often modeled as a S-MJLS. To study the behavior of network control systems in a finite interval, the investigation of the finite-time stability for S-MJLSs is necessary. Until now, the pursuing of less conservative controller design methods for S-MJLSs is still difficult in solving the addressed problem. Although there are some existing results that can guarantee the finite-time stability or finite-time boundedness of the system,[23-25] it is worth noting that the results of finite-time stability for S-MJLSs with time delay so far are still conservative and leave much room for further improvement, which motivates our present study.

In this paper, our purpose is to solve the problems of finite-time stochastic stability and exponential stochastic stability analysis and stabilization for continuous-time S-MJLSs with time-varying delay. At first, a new Lyapunov-Krasovskii functional is proposed, then the new finite-time stochastic stability criterion is also proposed. Meanwhile, the stability criterion is proved to guarantee the exponential stochastic stability of the system. Furthermore, the stabilization criterion is also proposed for S-MJLSs with time delay. Finally, Example 1 is used to demonstrate the less conservatism of the developed results than that of other existing results. Furthermore, we use a one-area load frequency control (LFC) to illustrate the effectiveness of the presented results as in Example 2.

The contributions of this paper can be summarized as follows. First, the result is presented to guarantee the finite-time stochastic stability of the system by using a new Lyapunov-Krasovskii functional combined with Wirtinger-based integral inequality. It is worth noting that by using Wirtinger-based integral inequality, the proposed Lyapunov-Krasovskii functional needs to be well designed to deal with the extra terms. Second, part of the conditions concerned with the finite-time stochastic stability of the system can also guarantee the stochastic stability of the system. Thus, using the advanced triangle inequality and the new Lyapunov-Krasovskii functional, the result is less conservative than some existing results.

We organized the remainder of this paper as follows. The model of S-MJLSs with time-varying delay is introduced, and the main problems are described in Section 2. In Section 3, a new stability criterion is derived by a new Lyapunov functional and a new triangle inequality. Furthermore, the stabilization criterion is also proposed. We use an LFC system to illustrate the effectiveness of our proposed methods in Section 4. Finally, we conclude this paper in Section 5.

Notation. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the set of real n-vector and $m \times n$ matrices, respectively. The superscript T stands for matrix transpose. The notation $P > 0(\geq 0)$ means that matrix P is positive definite or semi-positive definite. I_n denotes an identity matrix with dimension n, and $0_{m,n}$ denotes an $m \times n$ dimension zero matrix. "$*$" is used to denote the symmetric terms in a block matrix P, $\{P\}_i$ represents the ith row of its explicitly expressed block structure, sym(P) is short for $P + P^T$, diag$\{\cdots\}$ means a block diagonal matrix, and $\mathcal{S}$ denotes the set $\{1, 2, \dots, s\}$.

2 | PROBLEM FORMULATION

In this section, we introduce the model of S-MJLSs with time-varying delay. Then, some definitions and lemmas are introduced for the subsequent development. Consider the following class of S-MJLSs with time-varying and mode-dependent delays:

$$\begin{aligned} &\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x\left(t - \tau_{r(t)}(t)\right) + B(r(t))u(t), \\ &x(t) = \psi(t), t \in [-\tau_M, 0],\ r(0) = r_0, \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}^p$ is the control input, $x(t) \in \mathbb{R}^n$ is the state vector, and $\tau_{r(t)}(t) \in [\tau_m, \tau_M]$ is assumed to be time varying and mode dependent. $\psi(t)$ is the given initial condition, which is a continuous function defined on the interval $[-\tau_M, 0]$. $r(t)$ is a finite-state Markovian jump process representing the system mode, which takes discrete values in a given finite set $\mathcal{S} = \{1, 2, 3, \dots, s\}$, and r_0 is the initial mode.

Defining $\bar{\Pi} = [\pi_{ij}(h)]$, where $i, j = 1, 2, \dots, s$, the evolution of the semi-Markov process $\{r(t), t \geqslant 0\}$ is governed by the following probability transitions:

$$\Pr\{r(t+h) = j | r(t) = i\} = \begin{cases} \pi_{ij}(h)h + o(h) & i \neq j \\ 1 + \pi_{ii}(h)h + o(h) & i = j, \end{cases} \tag{2}$$

where $\pi_{ij}(h)$ is the transition rate from mode i to mode j at t, for $i \neq j$, with

$$\sum_{j=1, j\neq i}^{s} \pi_{ij}(h) = -\pi_{ii}(h), \tag{3}$$

for each mode $i, j \in \mathcal{S}$, $o(\Delta t)/\Delta t \to 0$. $A(r(t))$, $A_d(r(t))$, and $B(r(t))$ are denoted as A_i, A_{di}, and B_i, where i is the mode of systems. $\tau_i(t)$ denotes the mode-dependent time-varying state delay in the system and satisfies the following condition:

$$\begin{aligned} 0 \leq \tau_{mi} \leq \tau_i(t) \leq \tau_{Mi}, \\ \dot{\tau}_i(t) \leq u, \end{aligned}$$

with $\tau_m = \min_{i\in\mathcal{S}} \tau_{mi}$ and $\tau_M = \min_{i\in\mathcal{S}} \tau_{Mi}$.

First, we introduce the following definitions and lemmas.

Define the transition rate matrix in the Markov process as follows:

$$\bar{\Pi} = \begin{bmatrix} \pi_{11}(h) & \pi_{12}(h) & \cdots & \pi_{1s}(h) \\ \pi_{21}(h) & \pi_{22}(h) & \cdots & \pi_{2s}(h) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{s1}(h) & \pi_{s2}(h) & \cdots & \pi_{ss}(h) \end{bmatrix}. \tag{4}$$

Remark 1. Huang and Shi[13] used Weibull distribution of sojourn time to replace the exponential distribution of sojourn time, and then, the transition rate in S-MJLSs is time varying instead of constant in MJLSs. In practice, the transition rate $\pi_{ij}(h)$ is generally bounded by $\underline{\pi}_{ij} \leqslant \pi_{ij}(h) \leqslant \bar{\pi}_{ij}$, where $\underline{\pi}_{ij}$ and $\bar{\pi}_{ij}$ are the lower and upper bound of $\pi_{ij}(h)$. In our paper, we use $\pi_{ij}(h) = \pi_{ij} + \Delta\pi_{ij}$ to describe $\pi_{ij}(h)$, where $\pi_{ij} = \frac{1}{2}(\underline{\pi}_{ij} + \bar{\pi}_{ij})$ and $|\Delta\pi_{ij}| \leqslant \kappa_{ij}$, $\kappa_{ij} = \frac{1}{2}(\bar{\pi}_{ij} - \underline{\pi}_{ij})$.

Definition 1. (See the work of Cheng et al[26])
The autonomous system (1) is said to be finite-time stochastically stable (FTSS) with respect to $(c_1, c_2, T, \hat{R})$ if

$$\sup_{-\tau_M \leq \upsilon \leq 0} E\left\{x^T(\upsilon)\hat{R}x(\upsilon), \dot{x}^T(\upsilon)\hat{R}\dot{x}(\upsilon)\right\} \leq c_1 \Rightarrow E\left\{x^T(t)\hat{R}x(t)\right\} < c_2,\ \forall t \in [0, T],$$

where $\hat{R} > 0$ and c_1 and c_2 are 2 positive scalars with $c_2 > c_1$.

Definition 2. (See the work of Gao et al[6])
For any finite $\psi(t) \in \mathbb{R}^n$ defined on $[-\tau_M, 0]$ and initial mode $r_0 \in S$, the S-MJLS in (1) is exponentially stochastically stable if there exist positive constants ϵ and α such that

$$E|x(t)|^2 \leq \epsilon e^{-\alpha t} \, |\psi|^2_{\tau_M},$$

where $|\psi|_{\tau_M} = \sup_{-\tau_M \leq s \leq 0} |\psi(s)|$ for any possible continuous ψ. ϵ is a decay coefficient and α is a decay rate.

Lemma 1. (See the work of Seuret and Gouaisbaut[27])
For any matrix $R > 0$ and a differentiable signal x in $[\alpha, \beta] \rightarrow R^n$, the following inequality holds:

$$-\int_\alpha^\beta \dot{x}^T(s) R \dot{x}(s) ds \leq \frac{1}{\beta - \alpha} \varpi^T \hat{\Omega} \varpi,$$

where

$$\hat{\Omega} = \begin{bmatrix} -4R & -2R & 6R \\ * & -4R & 6R \\ * & * & -12R \end{bmatrix}, \; \varpi = \begin{bmatrix} x^T(\beta) & x^T(\alpha) & \frac{1}{\beta-\alpha} \int_\alpha^\beta x^T(s) ds \end{bmatrix}^T.$$

Lemma 2. (See the work of Li et al[17])
Given any scalar ε and square matrix $H \in \mathbb{R}^{n \times n}$, the following inequality, ie,

$$\varepsilon(H + H^T) \leq \varepsilon^2 T + H T^{-1} H^T,$$

holds for any symmetric positive definite matrix $T \in \mathbb{R}^{n \times n}$.

According to the above definitions and lemmas, the following problems are addressed.

1. Stability analysis: we present new conditions to guarantee the finite-time stochastic stability of the open-loop system.
2. Stabilization: we propose a new controller design method on the basis of the new criterion to guarantee the finite-time stochastic stability of the closed-loop system.

3 | MAIN RESULTS

In this section, we discuss the finite-time stochastic stability for the system in (1). Furthermore, the relationship between the finite-time stochastic stability and the exponential stochastic stability for the time-delay S-MJLSs is proved by a corollary. Lastly, a new controller design criterion is also proposed.

3.1 | Stability analysis for time-delay semi-Markovian jump systems

In this section, we present a finite-time stability criterion according to a new triangle inequality and a new Lyapunov-Krasovskii functional.

Theorem 1. *The S-MJLS in (1) with $u(t) = 0$ is FTSS with respect to $(c_1, c_2, T, \hat{R})$ if there exists a set of matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $Q_{3i} > 0$, $S_1 > 0$, $S_2 > 0$, $R_1 > 0$, $R_2 > 0$, M_i such that the following inequalities hold for all $i \in S$:*

$$\Phi_{1i} < 0, \tag{5}$$

$$\Phi_{2i} < 0, \tag{6}$$

$$e^{\alpha\tau_m} \sum_{j=1}^{s} \pi_{ij}(h) Q_{1j} - e^{\alpha\tau_M} \sum_{j=1, i\neq j}^{s} \pi_{ij}(h) Q_{2j} - S_1 < 0, \tag{7}$$

$$e^{\alpha\tau_M} \sum_{j=1, i\neq j}^{s} \pi_{ij}(h) Q_{2j} + e^{\alpha\tau_M} \sum_{j=1}^{s} \pi_{ij}(h) Q_{3j} - S_2 < 0, \tag{8}$$

$$e^{\alpha\tau_M} \sum_{j=1}^{s} \pi_{ij}(h) Q_{3j} - S_2 < 0, \tag{9}$$

$$c_1 \Lambda < \lambda_1 e^{-\alpha T} c_2, \tag{10}$$

where

$$
\begin{aligned}
\Phi_i = {} & \Pi_2^T\left(\sum_{j=1}^{s} P_j\pi_{ij}(h) + \alpha P_i\right)\Pi_2 + \mathrm{sym}\left(\Pi_1^T P_i \Pi_2\right) + \mathrm{sym}\left(\Pi_5^T M_i \omega_i \Pi_5\right) \\
& + \Sigma_1^T\left[e^{\alpha\tau_m}Q_{1i} + e^{\alpha\tau_M}Q_{3i} + \frac{e^{\alpha\tau_m}-1}{\alpha}S_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}S_2\right]\Sigma_1 + \Sigma_2^T\left(-Q_{1i} + e^{\alpha\left(\tau_M-\tau_m\right)}Q_{2i}\right)\Sigma_2 \\
& + \Sigma_3^T\left(\mu e^{\alpha(\tau_M-\tau_m)} - 1\right)Q_{2i}\Sigma_3 - \Sigma_4^T Q_{3i}\Sigma_4 + \Sigma_5^T\left(\frac{e^{\alpha\tau_m}-1}{\alpha}R_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}R_2\right)\Sigma_5 + \Pi_3^T\frac{1}{\tau_m}\Omega_1\Pi_3, \\
\Phi_{1i} = {} & \Phi_i + \Pi_4^T\frac{2}{\tau_M}\Omega_2\Pi_4 + \frac{1}{\tau_M}\Pi_6^T\Omega_2\Pi_6,\ \Phi_{2i} = \Phi_i + \Pi_4^T\frac{1}{\tau_M}\Omega_2\Pi_4 + \frac{2}{\tau_M}\Pi_6^T\Omega_2\Pi_6, \\
\Pi_1 = {} & \left[\Sigma_5^T \quad \Sigma_1^T - \Sigma_2^T \quad \Sigma_1^T - (1-u)\Sigma_3^T\right]^T,\ \Pi_2 = \left[\Sigma_1^T \quad \tau_m\Sigma_6^T \quad \tau_M\Sigma_7^T\right]^T, \\
\Pi_3 = {} & \left[\Sigma_1^T \quad \Sigma_2^T \quad \Sigma_6^T\right]^T,\ \Pi_4 = \left[\Sigma_1^T \quad \Sigma_4^T \quad \Sigma_7^T\right]^T,\ \Pi_5 = \left[\Sigma_1^T \quad \Sigma_2^T \quad \Sigma_3^T \quad \Sigma_4^T \quad \Sigma_5^T\right]^T, \\
\Pi_6 = {} & \left[\Sigma_3^T \quad \Sigma_4^T \quad \Sigma_8^T\right]^T,\ \Sigma_i = \left[0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(8-i)n}\right],\ i = 1, 2, \ldots, 8, \\
\Lambda = {} & 3\max_{i\in\mathcal{S}}\lambda_{\max}(\bar{P}_i) + \tau_M e^{\alpha\tau_M}\left(\max_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{1i}) + \max_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{2i}) + \max_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{3i})\right) \\
& + \tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}(\bar{R}_1) + \lambda_{\max}(\bar{R}_2)\right) + \tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}(\bar{S}_1) + \lambda_{\max}(\bar{S}_2)\right), \\
\bar{P}_i = {} & \mathrm{diag}\left\{\hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}}\right\} P_i \mathrm{diag}\left\{\hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}}\right\}, \\
\bar{Q}_{ni} = {} & \hat{R}^{-\frac{1}{2}}Q_{ni}\hat{R}^{-\frac{1}{2}},\ n = 1, 2, 3,\ \omega_i = \left[A_i \ 0 \ A_{di} \ 0 \ -I\right], \\
\bar{R}_m = {} & \hat{R}^{-\frac{1}{2}}R_m\hat{R}^{-\frac{1}{2}},\ \bar{S}_m = \hat{R}^{-\frac{1}{2}}S_m\hat{R}^{-\frac{1}{2}},\ m = 1, 2,\ \lambda_1 = \max_{i\in\mathcal{S}}\lambda_{\min}(\bar{P}_i), \\
\Omega_1 = {} & \begin{bmatrix} -4R_1 & -2R_1 & 6R_1 \\ * & -4R_1 & 6R_1 \\ * & * & -12R_1 \end{bmatrix}, \Omega_2 = \begin{bmatrix} -4R_2 & -2R_2 & 6R_2 \\ * & -4R_2 & 6R_2 \\ * & * & -12R_2 \end{bmatrix}.
\end{aligned}
$$

Proof. In the proof, we construct the stochastic Lyapunov-Krasovskii functional candidate as follows:

$$
V(x(t), r(t), t) = V_1(x(t), r(t), t) + V_2(x(t), r(t), t) + V_3(x(t), r(t), t) + V_4\left(x(t), r(t), t\right), \tag{11}
$$

$$
\begin{aligned}
V_1(x(t), t) = {} & e^{\alpha t}\gamma_{r(t)}^T(t)P(r(t))\gamma_{r(t)}(t), \\
V_2(x(t), r(t), t) = {} & \int_{t-\tau_m}^{t} e^{\alpha\left(s+\tau_m\right)}x^T(s)Q_1(r(s))x(s)ds + \int_{t-\tau_{r(t)}(t)}^{t-\tau_m} e^{\alpha\left(s+\tau_M\right)}x^T(s)Q_2(r(s))x(s)ds \\
& + \int_{t-\tau_M}^{t} e^{\alpha\left(s+\tau_M\right)}x^T(s)Q_3(r(s))x(s)ds, \\
V_3(x(t), r(t), t) = {} & \int_{-\tau_m}^{0}\int_{t+\theta}^{t} e^{\alpha(s-\theta)}\dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{\alpha(s-\theta)}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta, \\
V_4(x(t), r(t), t) = {} & \int_{-\tau_m}^{0}\int_{t+\theta}^{t} e^{\alpha(s-\theta)}x^T(s)S_1x(s)dsd\theta + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{\alpha(s-\theta)}x^T(s)S_2x(s)dsd\theta,
\end{aligned}
$$

$$
\gamma_{r(t)}(t) = \left[x^T(t) \ \int_{t-\tau_m}^{t} x^T(s)ds \ \int_{t-\tau_{r(t)}(t)}^{t} x^T(s)ds\right]^T. \tag{12}
$$

The weak infinitesimal operator ∇ of the stochastic process $\{\, x_t \ \ r_t \,\}, t \geq 0$, is given as

$$
\nabla V(x(t), r(t), t) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left[E\left\{V\left(x(t+\Delta), r(t+\Delta), t\right)\left|x_t, r_t\right.\right\} - V\left(x(t), r(t), t\right)\right]. \tag{13}
$$

For different $r_t = i$, we have the following equations:

$$\nabla V_1 = \alpha V_1 + e^{\alpha t} \lim_{\Delta\to 0}\frac{1}{\Delta}\left[\begin{array}{c} E\left[\gamma^T_{r(t+\Delta)}(t+\Delta)P(r(t+\Delta))\gamma_{r(t+\Delta)}(t+\Delta)\right] \\ -\gamma^T_{r(t)}(t)P(r(t))\gamma_{r(t)}(t)\end{array}\right] \tag{14}$$

$$= \alpha V_1 + e^{\alpha t}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\begin{array}{c}\sum_{j=1,j\neq i}^{s}\Pr\{r(t+\Delta)=j|r(t)=i\}\,\gamma_i^T(t+\Delta)P_j\gamma_j(t+\Delta) \\ +\Pr\{r(t+\Delta)=i|r(t)=i\}\,\gamma_i^T(t+\Delta)P_i\gamma_i(t+\Delta)-\gamma_i^T(t)P_i\gamma_i(t)\end{array}\right]$$

$$= \alpha V_1 + e^{\alpha t}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\begin{array}{c}\sum_{j=1,j\neq i}^{s}\frac{q_{ij}\left(G_i(h+\Delta)-G_i(h)\right)}{1-G_i(h)}\gamma_j^T(t+\Delta)P_j\gamma_j(t+\Delta) \\ +\frac{1-G_i(h+\Delta)}{1-G_i(h)}\gamma_i^T(t+\Delta)P_i\gamma_i(t+\Delta)-\gamma_i^T(t)P_i\gamma_i(t)\end{array}\right],$$

where h is the time elapsed when the system stays at mode i from the last jump, $G_i(t)$ is the cumulative distribution function (CDF) of sojourn time when the system remains in mode i, and q_{ij} is the probability density of the system jump from mode i to mode j. Given that Δ is small, we have

$$\gamma_i(t+\Delta) = \gamma_i(t) + \Delta\dot{\gamma}_i(t) + o(\Delta).$$

Then, the infinitesimal generator becomes

$$\nabla V_1 = \alpha V_1(x(t),r(t),t) + e^{\alpha t}\left(\begin{bmatrix}\gamma_i(t)\\ \dot{\gamma}_i(t)\end{bmatrix}^T \phi(i,h)\begin{bmatrix}\gamma_i(t)\\ \dot{\gamma}_i(t)\end{bmatrix}\right),$$

where

$$\phi(i,h) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left[\begin{array}{c}\sum_{j=1,j\neq i}^{s}\frac{q_{ij}\left(G_i(h+\Delta)-G_i(h)\right)}{1-G_i(h)}\begin{bmatrix}P_j & \Delta P_j\\ * & 0\end{bmatrix} \\ +\frac{1-G_i(h+\Delta)}{1-G_i(h)}\begin{bmatrix}P_i & \Delta P_i\\ * & 0\end{bmatrix}-\begin{bmatrix}P_i & 0\\ 0 & 0\end{bmatrix}\end{array}\right].$$

According to the property of the CDF, we have

$$\lim_{\Delta\to 0}\frac{(G_i(h+\Delta)-G_i(h))}{(1-G_i(h))\,\Delta} = \pi_i(h),\ \lim_{\Delta\to 0}\frac{1-G_i(h+\Delta)}{1-G_i(h)} = 1,\ \lim_{\Delta\to 0}\frac{(G_i(h+\Delta)-G_i(h))}{1-G_i(h)} = 0,$$

where $\pi_i(h)$ is the transition rate of the system jumping from mode i. Defining $\pi_{ij}(h) \triangleq q_{ij}\pi_i(h)$ for $i \neq j$ and $\pi_{ii}(h) \triangleq -\sum_{j=1,j\neq i}^{s}\pi_{ij}(h)$, we obtain

$$\nabla V_1 = e^{\alpha t}\left[\alpha\gamma_i^T(t)P_i\gamma_i(t) + \gamma_i^T(t)\left(\sum_{j=1}^{s}P_j\pi_{ij}(h)\right)\gamma_i(t) + 2\dot{\gamma}_i^T(t)P_i\gamma_i(t)\right].$$

On the other hand, we have

$$\begin{aligned}\nabla V_2 \leq\ & e^{\alpha(t+\tau_m)}x^T(t)Q_{1i}x(t) - e^{\alpha t}x^T(t-\tau_m)Q_{1i}x(t-\tau_m) + \sum_{j=1}^{s}\pi_{ij}(h)\int_{t-\tau_m}^{t}e^{\alpha(s+\tau_m)}x^T(s)Q_{1j}x(s)ds \\ & + e^{\alpha(t+\tau_M-\tau_m)}x^T(t-\tau_m)Q_{2i}x(t-\tau_m) - e^{\alpha t}x^T(t-\tau_i(t))Q_{2i}x(t-\tau_i(t)) \\ & + \mu e^{\alpha(t+\tau_M-\tau_m)}x^T(t-\tau_i(t))Q_{2i}x(t-\tau_i(t)) \\ & + \sum_{j=1}^{s}\pi_{ij}(h)\int_{t-\tau_j(t)}^{t-\tau_m}e^{\alpha(s+\tau_M)}x^T(s)Q_{2j}x(s)ds + e^{\alpha(t+\tau_M)}x^T(t)Q_{3i}x(t) \\ & - e^{\alpha t}x^T(t-\tau_M)Q_{3i}x(t-\tau_M) + \sum_{j=1}^{s}\pi_{ij}(h)\int_{t-\tau_M}^{t}e^{\alpha(s+\tau_M)}x^T(s)Q_{3j}x(s)ds,\end{aligned} \tag{15}$$

$$\nabla V_3 = e^{\alpha t}\left(\begin{array}{c}\dot{x}^T(t)R_1\dot{x}(t)\frac{e^{\alpha\tau_m}-1}{\alpha} - \int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \\ +\dot{x}^T(t)R_2\dot{x}(t)\frac{e^{\alpha\tau_M}-1}{\alpha} - \int_{t-\tau_M}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds\end{array}\right),$$

$$\nabla V_4 = e^{\alpha t}\left(\begin{array}{c}x^T(t)S_1x(t)\frac{e^{\alpha\tau_m}-1}{\alpha} - \int_{t-\tau_m}^{t}x^T(s)S_1x(s)ds \\ +x^T(t)S_2x(t)\frac{e^{\alpha\tau_M}-1}{\alpha} - \int_{t-\tau_M}^{t}x^T(s)S_2x(s)ds\end{array}\right).$$

Noticing $\pi_{ij}(h) \geq 0$ for $j \neq i$ and $\pi_{ii}(h) \leq 0$, then we have

$$\begin{aligned}
\sum_{j=1}^{N}\pi_{ij}(h)\int_{t-\tau_i(t)}^{t-\tau_m} e^{(s+\tau_M)}x^T(s)Q_{2j}x(s)ds \leq& \int_{t-\tau_M}^{t-\tau_m} e^{\alpha(s+\tau_M)}x^T(s)\left(\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j}\right)x(s)ds \\
=& \int_{t-\tau_M}^{t} e^{\alpha(s+\tau_M)}x^T(s)\left(\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j}\right)x(s)ds \\
&- \int_{t-\tau_m}^{t} e^{\alpha(s+\tau_M)}x^T(s)\left(\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j}\right)x(s)ds.
\end{aligned}$$

Suppose $\epsilon_i = \frac{\tau_i(t)}{\tau_M}$, we have the following equations:

$$\begin{aligned}
-\int_{t-\tau_i(t)}^{t}\dot{x}^T(s)R_2x(s)ds \;&\leq -\frac{1-\epsilon_i}{\tau_M}\int_{t-\tau_i(t)}^{t}\tau_i(t)\dot{x}^T(s)R_2x(s)ds - \frac{1}{\tau_M}\int_{t-\tau_i(t)}^{t}\tau_i(t)\dot{x}^T(s)R_2x(s)ds, \\
-\int_{t-\tau_M}^{t-\tau_i(t)}\dot{x}^T(s)R_2x(s)ds \;&\leq -\frac{1}{\tau_M}\int_{t-\tau_M}^{t-\tau_i(t)}(\tau_M-\tau_i(t))\,\dot{x}^T(s)R_2x(s)ds - \frac{\epsilon_i}{\tau_M}\int_{t-\tau_M}^{t-\tau_i(t)}(\tau_M-\tau_i(t))\,\dot{x}^T(s)R_2x(s)ds.
\end{aligned}$$

According to Lemma 1, one obtains

$$\begin{aligned}
&-\int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \leq \frac{1}{\tau_m}\varpi_1^T\Omega_1\varpi_1, \\
&-\int_{t-\tau_i(t)}^{t}\tau_i(t)\dot{x}^T(s)R_2\dot{x}(s)ds \leq \varpi_2^T\Omega_2\varpi_2, \\
&-\int_{t-\tau_M}^{t-\tau_i(t)}(\tau_M-\tau_i(t))\,\dot{x}^T(s)R_2\dot{x}(s)ds \leq \varpi_3^T\Omega_2\varpi_3,
\end{aligned}$$

where

$$\begin{aligned}
\varpi_1 &= \begin{bmatrix} x^T(t) & x^T(t-\tau_m) & \frac{1}{\tau_m}\int_{t-\tau_m}^{t}x^T(s)ds \end{bmatrix}^T, \\
\varpi_2 &= \begin{bmatrix} x^T(t) & x^T(t-\tau_i(t)) & \frac{1}{\tau_i(t)}\int_{t-\tau_i(t)}^{t}x^T(s)ds \end{bmatrix}^T, \\
\varpi_3 &= \begin{bmatrix} x^T(t-\tau_i(t)) & x^T(t-\tau_M) & \frac{1}{\tau_M-\tau_i(t)}\int_{t-\tau_M}^{t-\tau_i(t)}x^T(s)ds \end{bmatrix}^T.
\end{aligned}$$

On the other hand, for appropriate dimension matrix M_i, we have

$$2\zeta_i^T(t)M_i\left[A_ix(t)+A_{di}x(t-\tau_i(t))-\dot{x}(t)\right]=0.$$

□

In this part, without loss of generality, 2 cases are discussed as follows.

Case 1. Assume that the following inequalities hold:

$$\begin{aligned}
e^{\alpha\tau_m}\sum_{j=1}^{s}\pi_{ij}(h)Q_{1j} - e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} \;&\geq\; 0, \\
e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} + e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} \;&\geq\; 0, \\
e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} \;&\geq\; 0.
\end{aligned}$$

In addition, we have

$$\int_{t-\tau_m}^{t} x^T(s)e^{\alpha s}\left(e^{\alpha\tau_m}\sum_{j=1}^{s}\pi_{ij}(h)Q_{1j} - e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j}\right)x(s)ds$$

$$\leq \int_{t-\tau_m}^{t} x^T(s)e^{\alpha t}\left(e^{\alpha\tau_m}\sum_{j=1}^{s}\pi_{ij}(h)Q_{1j} - e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j}\right)x(s)ds,$$

$$\int_{t-\tau_i(t)}^{t} x^T(s)e^{\alpha s}\left(e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} + e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j}\right)x(s)ds$$

$$\leq \int_{t-\tau_i(t)}^{t} x^T(s)e^{\alpha t}\left(e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} + e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j}\right)x(s)ds,$$

$$\int_{t-\tau_M}^{t-\tau_i(t)} x^T(s)e^{\alpha s}\left(e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} - S_2\right)x(s)ds$$

$$\leq \int_{t-\tau_M}^{t-\tau_i(t)} x^T(s)e^{\alpha t}\left(e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} - S_2\right)x(s)ds.$$

Furthermore, we obtain

$$\nabla V \leq e^{\alpha t}\zeta_i^T(t)\left((1-\epsilon_i)\Phi_{1i} + \epsilon_i\Phi_{2i}\right)\zeta_i(t) + e^{\alpha t}\left[\begin{array}{c} \int_{t-\tau_m}^{t} x^T(s)\left(e^{\alpha\tau_m}\sum_{j=1}^{s}\pi_{ij}(h)Q_{1j} - e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} - S_1\right)x(s)ds \\ + \int_{t-\tau_i(t)}^{t} x^T(s)\left(e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} + e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} - S_2\right)x(s)ds \\ + \int_{t-\tau_M}^{t-\tau_i(t)} x^T(s)\left(e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} - S_2\right)x(s)ds \end{array}\right] \tag{16}$$

$$< 0,$$

where

$$\zeta_i(t) = \left[\begin{array}{c} x^T(t), x^T(t-\tau_m), x^T\left(t-\tau_i(t)\right), x^T(t-\tau_M), \dot{x}^T(t), \\ \frac{1}{\tau_m}\int_{t-\tau_m}^{t} x^T(s)ds, \frac{1}{\tau_i(t)}\int_{t-\tau_i(t)}^{t} x^T(s)ds, \frac{1}{\tau_M-\tau_i(t)}\int_{t-\tau_M}^{t-\tau_i(t)} x^T(s)ds \end{array}\right]^T.$$

Case 2. If Case 1 cannot hold, the following inequalities must hold:

$$e^{\alpha\tau_m}\sum_{j=1}^{s}\pi_{ij}(h)Q_{1j} - e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} < 0,$$

$$e^{\alpha\tau_M}\sum_{j=1,i\neq j}^{s}\pi_{ij}(h)Q_{2j} + e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} < 0,$$

$$e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}(h)Q_{3j} < 0.$$

It is easy to obtain that

$$\nabla V \leq e^{\alpha t}\zeta_i^T(t)\left((1-\epsilon_i)\Phi_{1i} + \epsilon_i\Phi_{2i}\right)\zeta_i(t) < 0.$$

Then, we have the following relation according to the above inequality:

$$E\left[\nabla V\left(x_t, r_t, t\right)\right] \leq E\left[\alpha V\left(x_t, r_t, t\right)\right].$$

We integrate the aforementioned inequality between 0 and t, multiply the above inequality by $e^{-\alpha t}$, and obtain the following:

$$e^{-\alpha t}E\left[V\left(x_t, r_t, t\right)\right] - E\left[V\left(x_0, r_0, 0\right)\right] \leq 0.$$

From (11), one can obtain

$$E\left[V\left(x_0,r_0,0\right)\right] \le \left[\begin{array}{c} \max\limits_{i\in\mathcal{S}}\lambda_{\max}(\bar{P}_i)+2\max\limits_{i\in\mathcal{S}}\lambda_{\max}(\bar{P}_i) \\ +\tau_M e^{\alpha\tau_M}\left(\max\limits_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{1i})+\max\limits_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{2i})+\max\limits_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_{3i})\right) \\ +\tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}(\bar{R}_1)+\lambda_{\max}(\bar{R}_2)\right) \\ +\tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}(\bar{S}_1)+\lambda_{\max}(\bar{S}_2)\right) \end{array}\right] \times \sup_{-\tau_M\le s\le 0}\left\{x^T(s)\hat{R}x(s),\dot{x}^T(s)\hat{R}\dot{x}(s)\right\}$$
$$\le c_1\Lambda.$$

On the other hand, according to (11), we have

$$E\left[V\left(x_t,r_t,t\right)\right] \geqslant E\left[e^{\alpha t}x^T(t)P_ix(t)\right] \geqslant \max_{i\in\mathcal{S}}\lambda_{\min}(\bar{P}_i)E\left[x^T(t)\hat{R}x(t)\right] = \lambda_1 E\left[x^T(t)\hat{R}x(t)\right].$$

Then, it can be derived that

$$E\left[x^T(t)\hat{R}x(t)\right] < \frac{1}{\lambda_1}E\left[V\left(x_t,r_t,t\right)\right] \le \frac{1}{\lambda_1}e^{\alpha t}E\left[V\left(x_0,r_0,0\right)\right] \le \frac{1}{\lambda_1}e^{\alpha t}c_1\Lambda < c_2.$$

According to Definition 1, the unforced system (1) is said to be FTSS with respect to $(c_1, c_2, T, \hat{R})$. The proof is completed. □

Remark 2. It is worth noting that Wirtinger-based integral inequality is a better triangle inequality than Jensen's inequality and the free-weight matrix method. However, inevitably, it can also generate a more complexity of LMI conditions. With the development of the computer technology, the CPU of the computer becomes faster and faster. Thus, it still makes sense to trade-off the complexity for a less conservative result.

Remark 3. It is known that the finite-time stability and the Lyapunov stability are independent concepts[28]; a system can be finite-time stable but not necessarily required to be Lyapunov stable and vice versa. However, for the time-delay S-MJLSs we are dealing with, finite-time stability and Lyapunov stability share some common conditions. In fact, by deleting the condition in (10), we would have the exponential stochastic stability for the S-MJLS in (1) directly. The detailed proof will be listed in Corollary 1.

Corollary 1. *The S-MJLS in (1) with $u(t) = 0$ is exponentially stochastically stable if there exists a set of matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $Q_{3i} > 0$, $S_1 > 0$, $S_2 > 0$, $R_1 > 0$, $R_2 > 0$ such that the inequalities in (5), (6)-(8), and (9) hold for all $i \in \mathcal{S}$.*

Proof. From the inequality in (16), we have

$$\nabla V < 0.$$

On the other hand, defining $d_1 \triangleq \max_{i\in\mathcal{S}}\{\|A_i\|\}$, $d_2 \triangleq \max_{i\in\mathcal{S}}\{\|A_{di}\|\}$, we have

$$|x(t)| \le \left[d_2\tau_M + 1\right]|\psi|_{\tau_M} + \int_0^t d_1\,|x(s)|\,ds,$$

when $0 \le t \le \tau_M$. According to Gronwall-Bellman lemma, we have

$$|x(t)| \le d|\psi|_{\tau_M},$$

where $d = (d_2\tau_M + 1)\,e^{d_1\tau_M}$. For any $-\tau_M \le t - \tau_i(t) \le -\tau_m$, $|x(t-\tau_i(t))| \le \max\{1, d\}|\psi|_{\tau_M} = d|\psi|_{\tau_M}$. When $t > \tau_M$, according to Dynkin's formula, we have

$$EV(x_t,i,t) = EV(x_{\tau_M},i,\tau_M) + E\int_0^{t-\tau_M}\nabla V(x_s,i,s)ds \le \Lambda\,|\psi|^2_{\tau_M}. \tag{17}$$

On the other hand, we have

$$E\left[V\left(x_t,r_t,t\right)\right] \ge \max_{i\in\mathcal{S}}\lambda_{\min}(P_i)e^{\alpha t}E|x(t)|^2. \tag{18}$$

Combining (17) and (18), we have

$$E\left[|x(t)|^2\right] \le \frac{\Lambda}{\max\limits_{i\in\mathcal{S}}\lambda_{\min}(P_i)}e^{-\alpha t}\,|\psi|^2_{\tau_M}, \tag{19}$$

when $t \geq \tau_M$. It is easy to prove that when $0 \leq t \leq \tau_M$, the inequality in (19) always holds with the same method. Thus, from Definition 2, we know that the S-MJLS in (1) is exponentially stochastically stable, which completes the proof. □

If the time delay is constant in the system, which means $\tau_m = \tau_M$, we would have the following corollary.

Corollary 2. *The autonomous S-MJLS in (1) with $u(t) = 0$ and $\tau_m = \tau_M$ is FTSS with respect to $(c_1, c_2, T, \hat{R})$ if there exist a set of matrices $P_i > 0$, $Q_i > 0$, $S > 0$, $R > 0$, $\bar{M}_i$ such that the following inequalities hold for all $i \in \mathcal{S}$:*

$$\begin{aligned} \bar{\Phi}_i &< 0, \\ e^{\alpha\tau}\sum_{j=1}^{s}\pi_{ij}(h)Q_j - S &< 0, \\ c_1\bar{\Lambda} &< \bar{\lambda}_1 e^{-\alpha T}c_2, \end{aligned}$$

where

$$\begin{aligned} \bar{\Phi}_i &= \bar{\Pi}_2^T\left(\sum_{j=1}^{s}P_j\pi_{ij}(h) + \alpha P_i\right)\bar{\Pi}_2 + \mathrm{sym}\left(\bar{\Pi}_1^T P_i \bar{\Pi}_2\right) + \Sigma_1^T\left[e^{\alpha\tau}Q_i + \frac{e^{\alpha\tau}-1}{\alpha}S\right]\Sigma_1 \\ &\quad -\Sigma_2^T Q_i \Sigma_2 + \Sigma_3^T\left(\frac{e^{\alpha\tau}-1}{\alpha}R\right)\Sigma_3 + \bar{\Pi}_3^T\frac{1}{\tau}\bar{\Omega}\bar{\Pi}_3 + \mathrm{sym}\left(\bar{\Pi}_5^T\bar{M}_i\bar{\omega}_i\bar{\Pi}_5\right), \\ \bar{\Pi}_1 &= \left[\Sigma_3^T \ \Sigma_1^T - \Sigma_2^T\right]^T, \ \bar{\Pi}_2 = \left[\Sigma_1^T \ \tau\Sigma_4^T\right]^T, \ \bar{\Pi}_3 = \left[\Sigma_1^T \ \Sigma_2^T \ \Sigma_4^T\right]^T, \\ \bar{\Pi}_5 &= \left[\Sigma_1^T \ \Sigma_2^T \ \Sigma_3^T\right]^T, \ \bar{\Omega} = \begin{bmatrix} -4R & -2R & 6R \\ * & -4R & 6R \\ * & * & -12R \end{bmatrix}, \ \bar{\omega}_i = \left[A_i \ A_{di} \ -I\right], \\ \bar{\Lambda} &= 2\max_{i\in\mathcal{S}}\lambda_{\max}(\bar{P}_i) + \tau_M e^{\alpha\tau_M}\max_{i\in\mathcal{S}}\lambda_{\max}(\bar{Q}_i) + \tau_M^2 e^{\alpha\tau_M}\lambda_{\max}(\bar{R}) + \tau_M^2 e^{\alpha\tau_M}\lambda_{\max}(\bar{S}), \\ \bar{P}_i &= \left[\hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}}\right]^T P_i \left[\hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}}\right], \ \bar{Q}_i = \hat{R}^{-\frac{1}{2}}Q_i\hat{R}^{-\frac{1}{2}}, \\ \bar{R} &= \hat{R}^{-\frac{1}{2}}R\hat{R}^{-\frac{1}{2}}, \ \bar{S} = \hat{R}^{-\frac{1}{2}}S\hat{R}^{-\frac{1}{2}}, \ \bar{\lambda}_1 = \max_{i\in\mathcal{S}}\lambda_{\min}(\bar{P}_i). \end{aligned}$$

Remark 4. Considering the time-varying transition rate term $\pi_{ij}(h)$, it is impossible for us to solve the inequalities in Theorem 1 by MATLAB. It is worth noting that there exist many ways to deal with this problem, such as the approach in the work of Li et al,[17] remark 5 in the work of Shen et al,[2] and the method in the work of Huang and Shi.[14] Considering the length of this paper, we only use the method in the work of Li et al[17] to further derive the following feasible theorem according to Remark 1.

Theorem 2. *For given scalars $\tau_M > 0$, $\tau_m > 0$, $\kappa_{ij} > 0$, the S-MJLS in (1) is FTSS with respect to $(c_1, c_2, T, \hat{R})$ if there exists a set of matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $Q_{3i} > 0$, $S_1 > 0$, $S_2 > 0$, $R_1 > 0$, $R_2 > 0$ and T_{ij} such that the following inequalities hold for all $i \in \mathcal{S}$:*

$$\begin{bmatrix} \hat{\Phi}_{1i} + \Pi_2^T\left(\sum_{j=1}^{s}P_j\pi_{ij} + \frac{1}{4}\sum_{j=1,i\neq j}^{s}\kappa_{ij}^2 T_{ij}\right)\Pi_2 & \Psi_i \\ * & -F_i \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \hat{\Phi}_{2i} + \Pi_2^T\left(\sum_{j=1}^{s}P_j\pi_{ij} + \frac{1}{4}\sum_{j=1,i\neq j}^{s}\kappa_{ij}^2 T_{ij}\right)\Pi_2 & \Psi_i \\ * & -F_i \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} \sum_{j=1}^{s}\pi_{ij}\bar{Q}_{1j} + \frac{1}{4}\sum_{j=1,i\neq j}^{s}\kappa_{ij}^2 J_{ij} - S_1 & \Psi_{1i} \\ * & -F_{1i} \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \sum_{j=1}^{s}\pi_{ij}\bar{Q}_{2i} + \frac{1}{4}\sum_{j=1,i\neq j}^{s}\kappa_{ij}^2 L_{ij} - S_2 & \Psi_{2i} \\ * & -F_{2i} \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} e^{\alpha\tau_M}\sum_{j=1}^{s}\pi_{ij}Q_{3i} + \frac{1}{4}\sum_{j=1,i\neq j}^{s}\kappa_{ij}^2 D_{ij} - S_2 & \Psi_{3i} \\ * & -F_{3i} \end{bmatrix} < 0, \tag{24}$$

$$c_1\Lambda < \lambda_1 e^{-\alpha T}c_2, \tag{25}$$

where

$$\begin{aligned}
\hat{\Phi}_i \triangleq\ & \Pi_2^T(\alpha P_i)\Pi_2 + \mathrm{sym}\left(\Pi_1^T P_i \Pi_2\right) + \Sigma_1^T\left[e^{\alpha\tau_m}Q_{1i} + e^{\alpha\tau_M}Q_{3i} + \frac{e^{\alpha\tau_m}-1}{\alpha}S_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}S_2\right]\Sigma_1 \\
& + \Sigma_2^T\left(-Q_{1i} + e^{\alpha(\tau_M-\tau_m)}Q_{2i}\right)\Sigma_2 + \Sigma_3^T\left(\mu e^{\alpha(\tau_M-\tau_m)} - 1\right)Q_{2i}\Sigma_3 - \Sigma_4^T Q_{3i}\Sigma_4 \\
& + \Sigma_5^T\left(\frac{e^{\alpha\tau_m}-1}{\alpha}R_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}R_2\right)\Sigma_5 + \Pi_3^T\frac{1}{\tau_m}\Omega_1\Pi_3 + \mathrm{sym}\left(\Pi_5^T M_i\omega_i\Pi_5\right),
\end{aligned}$$

$$\hat{\Phi}_{1i} = \hat{\Phi}_i + \Pi_4^T\frac{2}{\tau_M}\Omega_2\Pi_4 + \frac{1}{\tau_M}\Pi_6^T\Omega_2\Pi_6,\ \hat{\Phi}_{2i} = \hat{\Phi}_i + \Pi_4^T\frac{1}{\tau_M}\Omega_2\Pi_4 + \frac{2}{\tau_M}\Pi_6^T\Omega_2\Pi_6,$$

$$\Psi_i \triangleq \Pi_2^T\left[P_i - P_1\ \cdots\ P_i - P_{i-1}\ P_i - P_{i+1}\ \cdots\ P_i - P_N\right],$$

$$\mathcal{F}_i \triangleq \mathrm{diag}\left\{T_{i,1}\ \cdots\ T_{i,(i-1)}\ T_{i,(i+1)}\ \cdots\ T_{i,N}\right\},\ \omega_i = \left[A_i\ 0\ A_{di}\ 0\ -I\right],$$

$$\bar{Q}_{1j} = e^{\alpha\tau_m}Q_{1j} - e^{\alpha\tau_M}Q_{2j},\ \bar{Q}_{2j} = e^{\alpha\tau_M}Q_{2j} + e^{\alpha\tau_M}Q_{3j},$$

$$\Psi_{1i} = \left[\bar{Q}_{1i} - \bar{Q}_{11}\ \cdots\ \bar{Q}_{1i} - \bar{Q}_{1i-1}\ \bar{Q}_{1i} - \bar{Q}_{1i+1}\ \cdots\ \bar{Q}_{1i} - \bar{Q}_{1N}\right],$$

$$\Psi_{2i} = \left[\bar{Q}_{2i} - \bar{Q}_{21}\ \cdots\ \bar{Q}_{2i} - \bar{Q}_{2i-1}\ \bar{Q}_{2i} - \bar{Q}_{2i+1}\ \cdots\ \bar{Q}_{2i} - \bar{Q}_{2N}\right],$$

$$\Psi_{3i} = e^{\alpha\tau_M}\left[Q_{3i} - Q_{31}\ \cdots\ Q_{3i} - Q_{3i-1}\ Q_{3i} - Q_{3i+1}\ \cdots\ Q_{3i} - Q_{3N}\right],$$

$$\mathcal{F}_{1i} \triangleq \mathrm{diag}\left\{J_{i,1}\ \cdots\ J_{i,(i-1)}\ J_{i,(i+1)}\ \cdots\ J_{i,N}\right\},$$

$$\mathcal{F}_{2i} \triangleq \mathrm{diag}\left\{L_{i,1}\ \cdots\ L_{i,(i-1)}\ L_{i,(i+1)}\ \cdots\ L_{i,N}\right\},$$

$$\mathcal{F}_{3i} \triangleq \mathrm{diag}\left\{D_{i,1}\ \cdots\ D_{i,(i-1)}\ D_{i,(i+1)}\ \cdots\ D_{i,N}\right\},$$

where $\underline{\pi}_{ij} < \bar{\pi}_{ij}$ are the lower and upper bounds of the transition rate, respectively.

Proof. According to Schur complement lemma, the inequality in (20) is equivalent to the following inequality:

$$\bar{\Phi}_{1i} + \Pi_2^T\left[\sum_{j=1}^{s}P_j\pi_{ij} + \sum_{j=1,j\neq i}^{s}\left[\frac{1}{4}\kappa_{ij}^2 T_{ij} + \left(P_j - P_i\right)T_{ij}^{-1}\left(P_j - P_i\right)\right]\right]\Pi_2 < 0.$$

From Lemma 2, the previous inequality holds for all $\left|\Delta\pi_{ij}\right| \le \kappa_{ij}$ if there exist matrices $T_{ij}(i,j \in \mathcal{S})$ such that

$$\Phi_{1i} = \tilde{\Phi}_{1i} + \Pi_2^T\sum_{j=1}^{s}P_j\left(\pi_{ij} + \Delta\pi_{ij}\right)\Pi_2 < 0, \tag{26}$$

which is the inequality in (5). Using the similar idea, we have that the inequalities in (21)-(23) and (24) can guarantee the inequalities in (6), (7), (8), and (9). According to Theorem 1, the system in (1) is FTSS with respect to (c_1, c_2, R, T), which completes the proof. □

3.2 | Control synthesis for time-delay semi-Markovian jump systems

In this section, we focus on the control synthesis of time-delay S-MJLSs. The state-feedback controller is given as

$$u(t) = K_i x(t). \tag{27}$$

Then, we have the following closed-loop system:

$$\begin{aligned}
\dot{x}(t) &= (A_i + B_iK_i)x(t) + A_{di}x(t - \tau_i(t)) \\
x(t) &= \psi(t), t \in [-\tau_M, 0], r(0) = r_0.
\end{aligned}$$

For the controller design, we have the following theorem.

Theorem 3. *For given scalars $\tau_M \ge \tau_m \ge 0$, there exists a state-feedback controller in the form of (27) such that the closed loop is FTSS with respect to $(c_1, c_2, T, \hat{R})$ and exponentially stochastically stable for any time-varying delay $\tau_i(t)$ if there exists a set of matrices $\tilde{P}_{11,i} > 0$,$\tilde{Q}_{1i} > 0$, $\tilde{Q}_{2i} > 0$,$\tilde{Q}_{3i} > 0$, $\tilde{S}_1 > 0$,$\tilde{S}_2 > 0$, $\tilde{R}_1 > 0$,$\tilde{R}_2 > 0$,$\tilde{T}_{ij}$,$\tilde{J}_{ij}$, $\tilde{L}_{ij}$, $\tilde{D}_{ij}$ such that the following inequalities hold for all $i \in \mathcal{S}$:*

$$\begin{bmatrix} \hat{\Phi}_{1i} & \hat{\Psi}_i & \Pi_2^T W_{\Theta_i}^T \\ * & -\hat{\mathcal{F}}_i & 0 \\ * & * & -\Theta_i \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} \hat{\Phi}_{2i} & \hat{\Psi}_i & \Pi_2^T W_{\Theta_i}^T \\ * & -\hat{F}_i & 0 \\ * & * & -\Theta_i \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \sum_{j=1}^{s} \pi_{ij}\hat{Q}_{1j} + \frac{1}{4}\sum_{j=1,i\neq j}^{s} \kappa_{ij}^2 \tilde{J}_{ij} - \tilde{S}_1 & \hat{\Psi}_{1i} \\ * & -\hat{F}_{1i} \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} \sum_{j=1}^{s} \pi_{ij}\hat{Q}_{2i} + \frac{1}{4}\sum_{j=1,i\neq j}^{s} \kappa_{ij}^2 \tilde{L}_{ij} - \tilde{S}_2 & \hat{\Psi}_{2i} \\ * & -\hat{F}_{2i} \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} e^{\alpha\tau_M}\sum_{j=1}^{s} \pi_{ij}\tilde{Q}_{3i} + \frac{1}{4}\sum_{j=1,i\neq j}^{s} \kappa_{ij}^2 \tilde{D}_{ij} - \tilde{S}_2 & \hat{\Psi}_{3i} \\ * & -\hat{F}_{3i} \end{bmatrix} < 0, \tag{32}$$

$$c_1\hat{\Lambda} < \tilde{\lambda}_1 e^{-\alpha T} c_2, \tag{33}$$

where

$$\begin{aligned}
\hat{\Phi}_i \triangleq{}& \Pi_2^T \left(\alpha\tilde{P}_i + \tilde{P}_i\pi_{ii} + \frac{1}{4}\sum_{j=1,i\neq j}^{s} \kappa_{ij}^2 \tilde{T}_{ij} \right)\Pi_2 \\
&+ \mathrm{sym}\left(\Pi_1^T\tilde{P}_i\Pi_2\right) + \Sigma_2^T\left(-\tilde{Q}_{1i} + e^{\alpha(\tau_M-\tau_m)}\tilde{Q}_{2i}\right)\Sigma_2 \\
&+ \Sigma_1^T\left[e^{\alpha\tau_m}\tilde{Q}_{1i} + e^{\alpha\tau_M}\tilde{Q}_{3i} + \frac{e^{\alpha\tau_m}-1}{\alpha}\tilde{S}_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}\tilde{S}_2\right]\Sigma_1 \\
&+ \Sigma_3^T\left(\mu e^{\alpha(\tau_M-\tau_m)} - 1\right)\tilde{Q}_{2i}\Sigma_3 - \Sigma_4^T\tilde{Q}_{3i}\Sigma_4 \\
&+ \Sigma_5^T\left(\frac{e^{\alpha\tau_m}-1}{\alpha}\tilde{R}_1 + \frac{e^{\alpha\tau_M}-1}{\alpha}\tilde{R}_2\right)\Sigma_5 + \Pi_3^T\frac{1}{\tau_m}\tilde{\Omega}_1\Pi_3 + \mathrm{sym}\left(\Pi_5^T\tilde{M}_i\tilde{\omega}_i\Pi_5\right),
\end{aligned} \tag{34}$$

$$\hat{\Phi}_{1i} = \hat{\Phi}_i + \Pi_4^T\frac{2}{\tau_M}\tilde{\Omega}_2\Pi_4 + \Pi_6^T\tilde{\Omega}_2\Pi_6, \hat{\Phi}_{2i} = \hat{\Phi}_i + \Pi_4^T\frac{1}{\tau_M}\tilde{\Omega}_2\Pi_4 + 2\Pi_6^T\tilde{\Omega}_2\Pi_6,$$

$$\hat{\Psi}_i \triangleq \left[\, \tilde{P}_i - \tilde{P}_1 \ \cdots \ \tilde{P}_i - \tilde{P}_{i-1} \ \tilde{P}_i - \tilde{P}_{i+1} \ \cdots \ \tilde{P}_i - \tilde{P}_N \,\right],$$

$$\hat{F}_i \triangleq \mathrm{diag}\left\{ \tilde{T}_{i,1} \ \cdots \ \tilde{T}_{i,(i-1)} \ \tilde{T}_{i,(i+1)} \ \cdots \ \tilde{T}_{i,N} \right\},$$

$$\omega_i = \left[A_i\tilde{P}_{11.i} + B_iY_i \ \ 0 \ \ A_{di}\tilde{P}_{11,i} \ \ 0 \ \ -\tilde{P}_{11,i} \right],$$

$$\check{Q}_{1i} = e^{\alpha\tau_m}\tilde{Q}_{1i} - e^{\alpha\tau_M}\tilde{Q}_{2i}, \check{Q}_{2i} = e^{\alpha\tau_M}\tilde{Q}_{2i} + e^{\alpha\tau_M}\tilde{Q}_{3i}, \tilde{P}_i = \mathrm{diag}\left\{ \tilde{P}_{11,i} \ \tilde{P}_{11,i} \ \tilde{P}_{11,i} \right\},$$

$$\tilde{M}_i = [\, e_1I_n \ e_2I_n \ e_3I_n \ e_4I_n \ e_5I_n \,],$$

$$W_{\Theta_i} = \left[\sqrt{\pi_{i,1}}\tilde{P}_i \ \cdots \ \sqrt{\pi_{i,i-1}}\tilde{P}_i \ \sqrt{\pi_{i,i+1}}\tilde{P}_i \ \cdots \ \sqrt{\pi_{i,N}}\tilde{P}_i \right],$$

$$\hat{\Psi}_{1i} = \left[\check{Q}_{1i} - \check{Q}_{11} \ \cdots \ \check{Q}_{1i} - \check{Q}_{1i-1} \ \check{Q}_{1i} - \check{Q}_{1i+1} \ \cdots \ \check{Q}_{1i} - \check{Q}_{1N} \right],$$

$$\hat{\Psi}_{2i} = \left[\check{Q}_{2i} - \check{Q}_{21} \ \cdots \ \check{Q}_{2i} - \check{Q}_{2i-1} \ \check{Q}_{2i} - \check{Q}_{2i+1} \ \cdots \ \check{Q}_{2i} - \check{Q}_{2N} \right],$$

$$\hat{\Psi}_{3i} = \left[\tilde{Q}_{3i} - \tilde{Q}_{31} \ \cdots \ \tilde{Q}_{3i} - \tilde{Q}_{3i-1} \ \tilde{Q}_{3i} - \tilde{Q}_{3i+1} \ \cdots \ \tilde{Q}_{2i} - \tilde{Q}_{2N} \right],$$

$$\hat{F}_{1i} \triangleq \mathrm{diag}\left\{ \tilde{J}_{i,1} \ \cdots \ \tilde{J}_{i,(i-1)} \ \tilde{J}_{i,(i+1)} \ \cdots \ \tilde{J}_{i,N} \right\},$$

$$\hat{F}_{2i} \triangleq \mathrm{diag}\left\{ \tilde{L}_{i,1} \ \cdots \ \tilde{L}_{i,(i-1)} \ \tilde{L}_{i,(i+1)} \ \cdots \ \tilde{L}_{i,N} \right\},$$

$$\hat{F}_{3i} \triangleq \mathrm{diag}\left\{ \tilde{D}_{i,1} \ \cdots \ \tilde{D}_{i,(i-1)} \ \tilde{D}_{i,(i+1)} \ \cdots \ \tilde{D}_{i,N} \right\},$$

$$\Theta_i = \mathrm{diag}\left\{ \tilde{P}_1 \ \cdots \ \tilde{P}_{i-1} \ \tilde{P}_{i+1} \ \cdots \ \tilde{P}_N \right\},$$

$$\begin{aligned}
\Lambda ={}& 3\max_{i\in S}\lambda_{\max}\left(\check{P}_i\right) + \tau_M e^{\alpha\tau_M}\left(\max_{i\in S}\lambda_{\max}\left(\check{Q}_{1i}\right) + \max_{i\in S}\lambda_{\max}\left(\check{Q}_{2i}\right) + \max_{i\in S}\lambda_{\max}\left(\check{Q}_{3i}\right)\right) \\
&+ \tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}\left(\check{R}_1\right) + \lambda_{\max}\left(\check{R}_2\right)\right) + \tau_M^2 e^{\alpha\tau_M}\left(\lambda_{\max}\left(\check{S}_1\right) + \lambda_{\max}\left(\check{S}_2\right)\right),
\end{aligned}$$

$$\tilde{\lambda}_1 = \max_{i\in N}\lambda_{\min}(\check{P}_i), \check{P}_i = \mathrm{diag}\left\{ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \right\}\tilde{P}_i\,\mathrm{diag}\left\{ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \ \hat{R}^{-\frac{1}{2}} \right\},$$

$$\check{Q}_{ni} = \hat{R}^{-\frac{1}{2}}\tilde{Q}_{ni}\hat{R}^{-\frac{1}{2}}, n = 1, 2, 3, \check{R}_m = \hat{R}^{-\frac{1}{2}}\tilde{R}_m\hat{R}^{-\frac{1}{2}}, \check{S}_m = \hat{R}^{-\frac{1}{2}}\tilde{S}_m\hat{R}^{-\frac{1}{2}}, m = 1, 2.$$

The closed-loop controller gains are as follows:

$$K_i = Y_i \tilde{P}_{11,i}^{-1}.$$

Proof. Define the following matrices:

$$\begin{aligned}
\Gamma^i_{7n\times 7n} &= \mathrm{diag}\left\{ P_{11,i}\ P_{11,i}\ P_{11,i}\ P_{11,i}\ P_{11,i}\ P_{11,i}\ P_{11,i} \right\}, \\
H &= \mathrm{diag}\left\{ \Gamma^i_{7n\times 7n}\ I_{(N-1)n} \right\}, \tilde{P}_{11i} = P_{11i}^{-1}, \\
\tilde{Q}_{1i} &= \tilde{P}_{11,i} Q_{1i} \tilde{P}_{11,i}, \tilde{Q}_{2i} = \tilde{P}_{11,i} Q_{2i} \tilde{P}_{11,i}, \tilde{Q}_{3i} = \tilde{P}_{11,i} Q_{3i} \tilde{P}_{11,i}, \\
\tilde{R}_1 &= \tilde{P}_{11,i} R_1 \tilde{P}_{11,i}, \tilde{R}_2 = \tilde{P}_{11,i} R_2 \tilde{P}_{11,i}, \tilde{S}_1 = \tilde{P}_{11,i} S_1 \tilde{P}_{11,i}, \\
\tilde{S}_2 &= \tilde{P}_{11,i} S_2 \tilde{P}_{11,i}, \tilde{T}_{ij} = \tilde{P}_i T_{ij}^T \tilde{P}_i, \tilde{J}_{ij} = \tilde{P}_{11,i} J_{ij} \tilde{P}_{11,i}, \\
\tilde{L}_{ij} &= \tilde{P}_{11,i} L_{ij} \tilde{P}_{11,i}.
\end{aligned}$$

By performing a congruence transformation H to the inequality in (28), we get

$$\Psi_{1i} = H^T \begin{bmatrix} \tilde{\Phi}_{1i} + \Pi_2^T \left(\sum_{j=1}^{s} \tilde{P}_j \pi_{ij} + \frac{1}{4} \sum_{j=1, i\neq j}^{s} \kappa_{ij}^2 \tilde{T}_{ij} \right) \Pi_2 & \tilde{\Psi}_i \\ * & -\tilde{F}_i \end{bmatrix} H < 0,$$

where $\tilde{\Phi}_i$,$\tilde{\Psi}_i$,$\tilde{P}_j$, $\tilde{F}_i$, and $\tilde{T}_{ij}$ are defined in (34). According to Schur complement lemma, $\Psi_i < 0$ guarantees the inequality in (20). Using the similar idea, we have that the inequalities in (29) to (32) can guarantee the inequalities in (21) to (24). According to Theorem 2, the closed-loop system is FTSS with respect to $(c_1, c_2, T, \hat{R})$. The proof is completed. □

Remark 5. We adopt the method by defining e_1, e_2,e_3, e_4, and e_5 with the aim to obtain a tractable matrix condition. By carefully choosing these parameters, the conservatism of the system would be further reduced.

4 | ILLUSTRATIVE EXAMPLES

In this section, we use 2 examples to demonstrate the effectiveness and advantages of our methods. The first example is to show the less conservatism of our method than any other methods, and the second one is to illustrative the effectiveness of our controller procedure.

Example 1. We present a simple example of S-MJLSs like (1) with 2 modes borrowed from the work of Li et al[29]:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, A_{d1} = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -1 & 0 \\ -0.1 & -1 \end{bmatrix}, \bar{\Pi} = \begin{bmatrix} \pi_{11}(h) & \pi_{12}(h) \\ \pi_{21}(h) & \pi_{22}(h) \end{bmatrix},$$

with initial conditions $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $r_0 = 1$. In the simulation, the mode $r(t)$ jumps between 1 and 2. First, we only consider time constant delay S-MJLSs. The comparison results with other methods[6,30] are displayed in Table 1.

TABLE 1 Maximum delay bound τ via different methods

π_{11}	−0.2	−0.5	−1
Theorem 2 of Shu et al[30]	0.352	0.349	0.346
Theorem 2 of Gao et al[6] m=5	0.822	0.813	0.808
Corollary 2	0.848	0.843	0.841

TABLE 2 Maximum decay rate via different upper bounds τ_M via different methods

α	0.1	0.2	0.3	0.4
Theorem 1 of Gao et al[6] m=5	1.570	1.415	1.300	1.212
Theorem 1	1.769	1.525	1.363	1.245

Furthermore, set $\tau_m = 1$, $\pi_{11} = -0.2$, and $\pi_{22} = -0.3$, and the maximum delay bound τ_M is listed with different α in Table 2. In addition, to further compare with more results, set $\tau_m = 1, \mu = 0.5$, and $\alpha = 0$, and then Theorem 2 can reduce to guarantee the exponential stochastic stability of the system. Thus, we can use Theorem 2 to compare with results in other works[17,31-33] and in the work of Yue and Han,[34] which is listed in Table 3. From Tables 1 to 3, we can conclude that our result is less conservative than those in other works[6,17,30-33] and in the work of Yue and Han.[34] Second, we consider time-varying transition rates. The transition rates in the model are $\pi_{11}(h) \in (-2.2, -1.8)$, $\pi_{12}(h) \in (1.8, 2.2)$, $\pi_{21}(h) \in (2.6, 3.4)$, and $\pi_{22}(h) \in (-3.4, -2.6)$. Then, according to Remark 1, we have $\pi_{11} = -2$, $\pi_{12} = 2$, $\pi_{21} = 3$, $\pi_{22} = -3$, and $\kappa_{1j} = 0.2$, $\kappa_{2j} = 0.4$. We want to find the maximum time delay τ_M such that the feasible solution still exists. The obtained maximum time delay τ_M is listed in Table 4 with the result in the work of Li et al[17] with partitioning number $l = 6$ and Theorem 2. It is worth noting that the result in the aforementioned work[17] uses the delay partitioning method to further reduce the conservatism. However, in our result, to simplify the result complexity, we do not use it. Even though, with μ becoming bigger and bigger, our result still becomes less conservative than that in the work of Li et al.[17]

Example 2. For demonstrating the effectiveness of the proposed controller design method for Theorem 3, we consider the dynamic model of a one-area LFC system shown in Figure 1, which can be expressed as follows[12]:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bu(t) + F\Delta P_d,$$
$$y(t) = Cx(t),$$

TABLE 3 Maximum upper delay bound τ_M with $\tau_m = 1$ and $\mu = 0.5$

Different Results	Maximum Allowed τ_M
Theorem 3.1 of Boukas et al[31]	0.224
Theorem 1 of Xu et al[33]	1.471
Theorem 1 of Yue and Han[34]	1.660
Theorem 1 of Fei et al[32] m=5	1.753
Theorem 3.1 of Li et al[17] l=4	1.807
Theorem 2	2.307

TABLE 4 Maximum upper delay bound $\tau_M$with $\tau_m = 1$

μ	0.2	0.5	0.9
Theorem 3.2 of Li et al[17] l=3	4.00	2.07	1.44
Theorem 3.2 of Li et al[17] l=6	4.11	2.16	1.57
Theorem 2	3.545	2.307	2.014

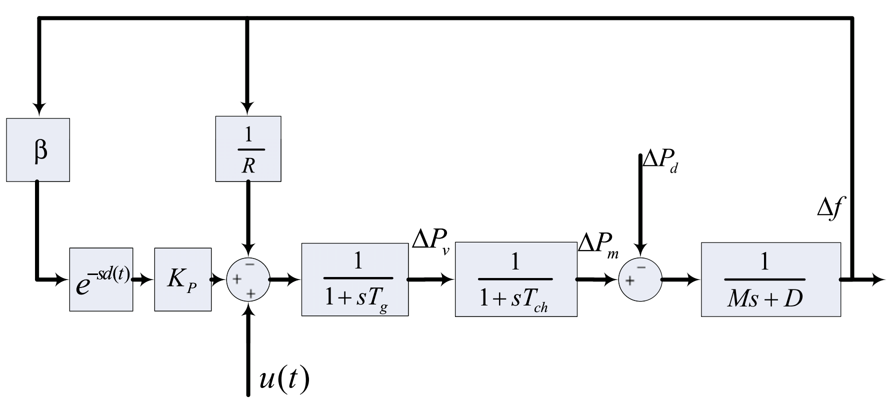

FIGURE 1 Dynamic model of one-area load frequency control scheme

where

$$x(t) = \begin{bmatrix} \Delta f & \Delta P_m & \Delta P_v \end{bmatrix}^T, B = \begin{bmatrix} 0 & 0 & \frac{1}{T_g} \end{bmatrix}^T, F = \begin{bmatrix} -\frac{1}{M} & 0 & 0 \end{bmatrix}^T$$

$$A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 \\ 0 & -\frac{1}{T_{ch}} & \frac{1}{T_{ch}} \\ -\frac{1}{RT_g} & 0 & -\frac{1}{T_g} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

where Δf is the deviation of frequency, ΔP_m is the generator mechanical output, and ΔP_v is the valve position. ΔP_d is the supply-demand mismatch disturbance. M,D,T_g, T_{ch}, and R denote the moment of inertia of the generator, generator damping coefficient, time constant of governor, time constant of the turbine, and speed drop, respectively. $K_p = 10$ is the power system gain. On the other hand, the parameters of LFC system always change due to the complex environment such as temperature and load fluctuation. Thus, S-MJLSs are considered with 2 modes (see Table 5) to describe the LFC system model accurately as follows:

$$A_1 = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -3.3 & -3.3 \\ -200 & 0 & -10 \end{bmatrix}, A_2 = \begin{bmatrix} -0.125 & 0.0833 & 0 \\ 0 & -2.5 & 2.5 \\ -117.65 & 0 & 5.88 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2100 & 0 & 0 \end{bmatrix},$$

$$A_{d2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1265 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 5.88 \end{bmatrix}, \bar{\Pi} = \begin{bmatrix} \pi_{11}(h) & \pi_{12}(h) \\ \pi_{21}(h) & \pi_{22}(h) \end{bmatrix},$$

with initial conditions $x(0) = \begin{bmatrix} 0.5 & 0.3 & 0.4 \end{bmatrix}^T$ and $r_0 = 1$. The transition rates in the model are $\pi_{11}(h) \in (-2.8, -3.2)$, $\pi_{12}(h) \in (2.8, 3.2)$, $\pi_{21}(h) \in (2.6, 3.4)$, and $\pi_{22}(h) \in (-2.6, -3.4)$. Then, according to Remark 1, we have $\pi_{11} = -3$, $\pi_{12} = 3$, $\pi_{21} = 3$, $\pi_{22} = -3$, and $\kappa_{1j} = 0.2$, $\kappa_{2j} = 0.4$. In the simulation, mode $r(t)$ jumps between 1 and 2. In this example, the stable interval of time delay can also be discussed, which guarantees the exponential stochastic stability of the closed-loop system. We design a state feedback in the form of (27) to make the closed-loop system exponentially stochastically stable. According to the above parameters and set $\alpha = 0.1$, $e_1 = 10$, $e_2 = 10$, $e_3 = 10$, $e_4 = 10$, $e_5 = 10$,

TABLE 5 Parameters of a load frequency control scheme

Parameter	$T_{ch}(s)$	$T_g(s)$	R	D	β	$M(s)$
Mode 1	0.3	0.1	0.05	1.0	21.0	10
Mode 2	0.4	0.17	0.05	1.5	21.5	12

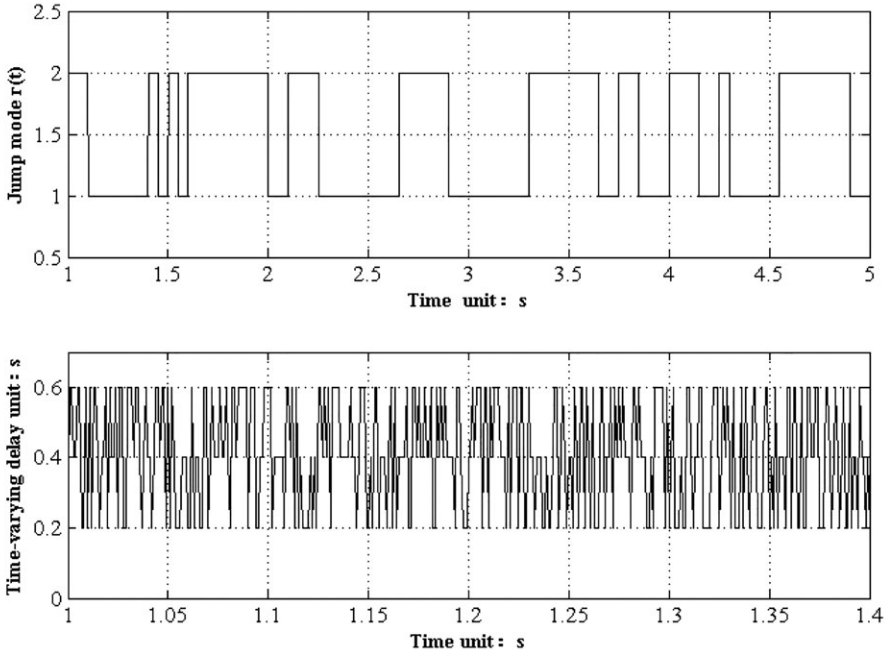

FIGURE 2 Time-varying delay and jump mode

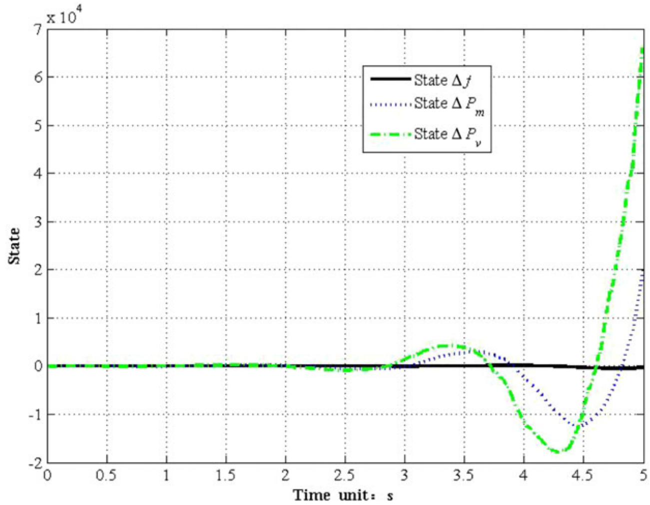

FIGURE 3 State response of the open-loop system [Colour figure can be viewed at wileyonlinelibrary.com]

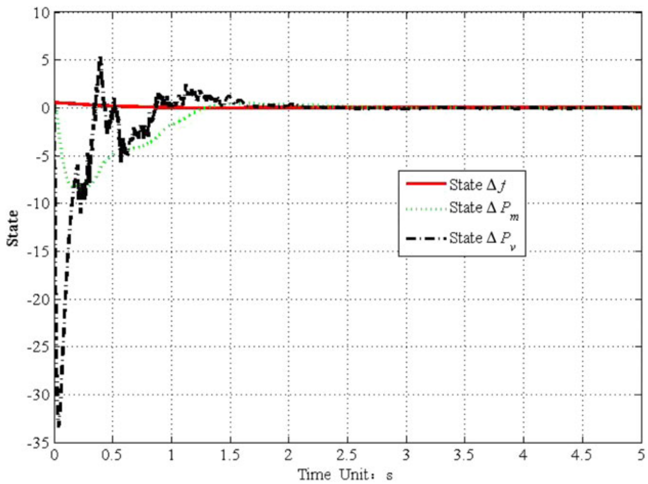

FIGURE 4 State response of the closed-loop system [Colour figure can be viewed at wileyonlinelibrary.com]

we can get the state-feedback controller gains as follows:

$$K_1 = \begin{bmatrix} -455.1933 & -18.4575 & -4.7612 \end{bmatrix}, K_2 = \begin{bmatrix} -242.4706 & -18.3610 & -5.8928 \end{bmatrix}. \tag{35}$$

Assume $\hat{R} = I$, we can get that $\hat{\Lambda} = 0.0054$, $\hat{\lambda}_1 = 1.29 \times 10^{-12}$. Set $T = 5$, according to the inequality in (10), we have $0.0054c_1 < 7.8 \times 10^{-13}c_2$. Thus, set $c_1 = 1$ and $c_2 = 6.9 \times 10^9$, which can guarantee the finite-time stability of the system. Figure 2 shows the jump mode $r(t)$ and the time delay in (1). Figure 3 shows the performance of states for the open-loop S-MJLSs. From Figure 3, we know that the open-loop system is unstable. If the state-feedback controller obtained by using the controller in (35), the closed-loop system becomes stable, which is shown in Figure 4. From the Figures, we conclude that the controller design method is effective.

5 | CONCLUSION

The stability analysis and stabilization for S-MJLSs with time delay have been studied in this paper. According to a new Lyapunov-Krasovskii functional, we have discussed the finite-time stability and exponential stochastic stability of the systems. Some new criteria about finite-time stochastic stability and exponential stochastic stability have been proposed. Part of conditions in the new finite-time stochastic stability criterion has also been proved to guarantee the exponential stochastic stability of the system. Meanwhile, the stabilization criterion has been proposed on the basis of the stability criterion such that the closed-loop system is FTSS. Lastly, Example 1 has been given to show the advantages of the stability criterion, and in Example 2, we have introduced the controller design process of a load frequency system and illustrate the effectiveness of the proposed controller design method. Since the structure of LFC system in Example 2 is the simplest one, our future work is to focus on the controller design of the more complex LFC system.

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